M. Gross – 27/5/08 – The tropical vertex (w/ Pandharipande + Siebert)

1. The tropical vertex group:

$M \simeq \mathbb{Z}^2$ rank 2 lattice, $N \simeq \mathrm{Hom}(M, \mathbb{Z})$ dual lattice.

k = char. 0 field (e.g. $\mathbb{Q}$)

R = complete local k-alg. with max. ideal $\mathfrak{m}$ (e.g. power series)

Define a subgroup $H(R) \subseteq \mathrm{Aut}(k[M] \hat{\otimes}_k R)$

where instanton correction transformations "live".

Def: (cf. Kontsevich & Soibelman) $H(R)$ = subgroup of $\mathrm{Aut}(k[M] \hat{\otimes}_k R)$ generated by automorphisms of the form $z^m \mapsto z^m f^{(n_0, m)}$ (& id on R)

where
$$\begin{cases} n_0 \in N \\ f \in k[z^{m_0}] \hat{\otimes}_k R \subseteq k[M] \hat{\otimes}_k R \text{ for some nonzero } m_0 \in M \\ f - 1 \in z^{m_0} \mathfrak{m} \quad (\Rightarrow \text{invertibility of } f) \\ \langle n_0, m_0 \rangle = 0. \end{cases}$$

Rmk: • elements of $H(R)$ are symplectomorphisms, preserving the sympl. form $\Omega = \frac{dx}{x} \wedge \frac{dy}{y}$ (because of condition $\langle n_0, m_0 \rangle = 0$).

Ex: $R = k[[t]]$, $\quad \begin{array}{l} x \mapsto x \\ y \mapsto y(1 + tx) \end{array}$ is a typical elt of $H(R)$

(here $m_0 = (1,0)$, $n_0 = (0,1)$).

2. Scattering diagrams

Def: A *ray* is a pair $(\mathfrak{d}, f_{\mathfrak{d}})$, $\mathfrak{d} \subseteq M_{\mathbb{R}}$ given by

$$\mathfrak{d} = m_0' - \mathbb{R}_{\geq 0} m_0 \quad (m_0' \in M_{\mathbb{R}},\ m_0 \in M \setminus \{0\})$$

ray in direction of m_0

and $f_{\mathfrak{d}} \in k[z^{m_0}] \hat{\otimes}_k R$ s.t. $f_{\mathfrak{d}} - 1 \in z^{m_0} \mathfrak{m}$

Def: A *line* = same thing except $\mathfrak{d} = m_0' - \mathbb{R} m_0$

instead of $\mathbb{R}_{\geq 0}$

Def: A *scattering diagram* $\mathfrak{D}$ = collection of rays & lines

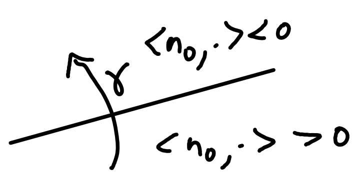
Consider $\gamma : [0,1] \to M_{\mathbb{R}}$ path

- transverse to every element of $\mathfrak{D}$
- doesn't pass through intersections of rays/lines nor endpoints of rays
- crosses each elt of $\mathfrak{D}$ only finitely many times

$\rightsquigarrow$ to such γ associate a path-ordered product of autom's.

- When γ crosses $(\mathfrak{d}, f_{\mathfrak{d}})$, get $z^m \mapsto z^m f_{\mathfrak{d}}^{\langle m, n_0 \rangle} \in H(R)$

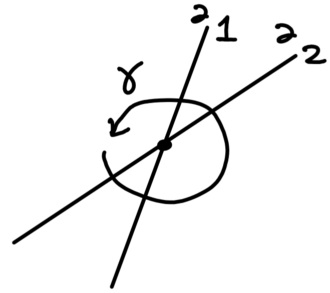
  where $n_0 \in N$ = primitive normal to ray



$$\begin{cases} \langle n_0, m_0 \rangle = 0 \\ n_0 \text{ primitive} \\ \langle n_0, \dot{\gamma}(t_0) \rangle < 0. \end{cases}$$

Then define $\boxed{\theta_{\mathcal{D},\gamma} = \prod_{\text{crossings}} \theta_i}$ ordered along γ.

<u>Example</u>:



$$\mathcal{D} = \{(\partial_1, f_1), \ (\partial_2, f_2)\}$$

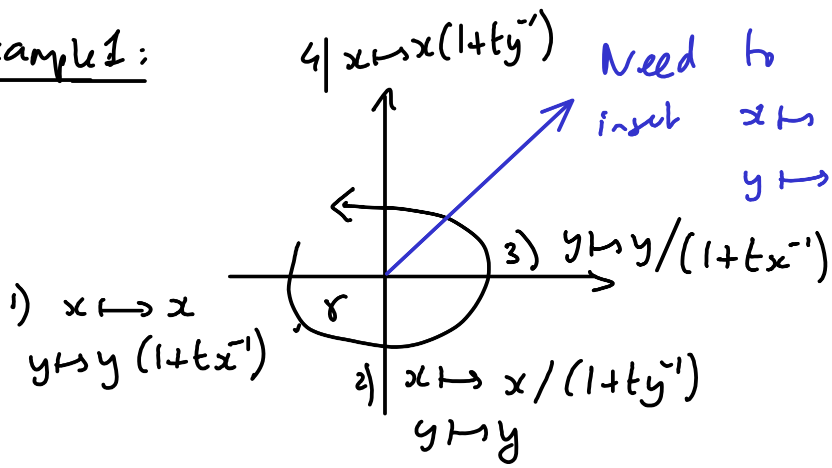
$$\rightsquigarrow \theta_{\mathcal{D},\gamma} = \theta_1 \theta_2 \theta_1^{-1} \theta_2^{-1}$$

<u>Kontsevich-Soibelman lemma</u>:

Given scattering diagram $\mathcal{D}$, $\exists \ \mathcal{D}' \supset \mathcal{D}$ s.t. $\mathcal{D}' \setminus \mathcal{D}$ consists only of <u>rays</u>, and $\theta_{\mathcal{D}',\gamma} = id$ for every closed loop γ s.t. $\theta_{\mathcal{D}',\gamma}$ is defined. [$\mathcal{D}'$ is essentially unique]

(<u>Pf</u>: cancel <u>order by order</u> the monodromies at crossings. Can't cancel a line by a ray, but can cancel commutators ... It's algorithmic.).

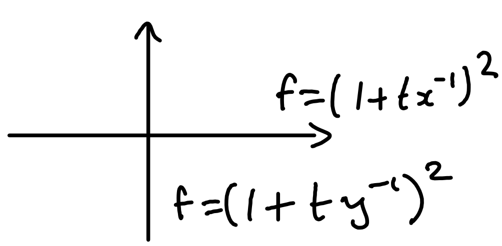
<u>Example 1</u>:



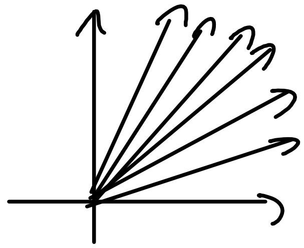
Need to insert $x \mapsto x(1+t^2x^{-1}y)$
$y \mapsto y(1+t^2x^{-1}y)$

A single ray cancels monodromy

<u>Example 2</u>:



$\Rightarrow$ need ∞ rays!

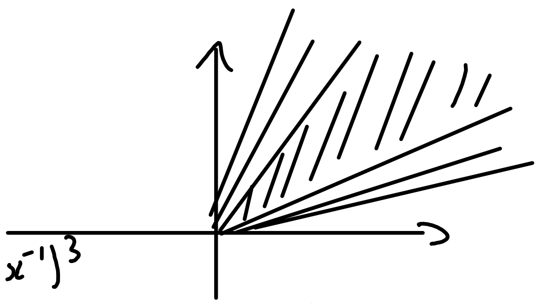


rays of slopes $\frac{n+1}{n}$ and $\frac{n-1}{n}$

$$\text{rays} \begin{cases} (n+1)/n : & (1+t^{2n+1}x^{-n}y^{-n-1})^2 \\ (n-1)/n : & (1+t^{2n+1}x^{-n-1}y^{-n})^2 \\ \text{slope } 1 : & (1-t^2x^{-1}y^{-1})^{-4} = \dfrac{(1+t^2x^{-1}y^{-1})^4}{(1-t^4x^{-2}y^{-2})^{2\cdot 2}} \end{cases}$$

(Algorithm works by ↑ order in t, so basically does them in increasing n's).

<u>Ex. 3:</u>



get rays of slope

$\frac{3}{8}, \frac{8}{21}$, etc... $\to \frac{3-\sqrt{5}}{2}$

$\frac{8}{3}, \frac{21}{8}$, etc... $\to \frac{3+\sqrt{5}}{2}$

and <u>all</u> rational slopes between these two limits!!

e.g. slope 1 ray is
$$\left(\sum_{n=0}^{\infty} \frac{1}{3n+1} \binom{4n}{n} t^{2n} x^{-n} y^{-n} \right)^9$$

$$= \frac{(1+t^2x^{-1}y^{-1})^9 (1+t^6x^{-3}y^{-3})^{3\cdot 54} \cdots}{(1-t^4x^{-2}y^{-2})^{2\cdot 18} (1-t^8x^{-4}y^{-4})^{4\cdot 252} \cdots}$$

The exponents have <u>enumerative interpretation</u>!